Classification of irreducible Harish-Chandra modules over generalized Virasoro algebras[1]

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Abstract

Let G be an arbitrary additive subgroup of $\mathbb{C}$ and $\mathrm{Vir}[G]$ the corresponding generalized Virasoro algebra. In the present paper, irreducible weight modules with finite dimensional weight spaces over $\mathrm{Vir}[G]$ are completely determined. The classification strongly depends on the index group G. If G does not have a direct summand $\mathbb{Z}$, then such irreducible modules over $\mathrm{Vir}[G]$ are only modules of intermediate series whose weight spaces are all 1-dimensional. Otherwise, there is one more class of modules which are constructed by using intermediate series modules over a generalized Virasoro subalgebra $\mathrm{Vir}[G_0]$ of $\mathrm{Vir}[G]$ for a direct summand G_0 of G with corank 1.

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§1. Introduction

The Virasoro algebra theory has been widely used in many physics areas and other mathematical branches, for example, string theory, 2-dimensional conformal field theory, differential geometry, combinatorics, Kac-Moody algebras, vertex algebras, and so on.

The generalized Virasoro algebras were first introduced and studied by mathematicians and mathematical physicists J. Patera and H. Zassenhaus [PZ] in 1991. Because of its own interest and the close relation between the theory of generalized Virasoro algebras and physics, this theory has attracted extensive attentions of mathematicians and physicists, particularly, the representation theory of generalized Virasoro algebras has been developed rapidly in the last decade. Let us first recall the definitions of these Lie algebras.

In this paper we denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the set of complex numbers, integers, nonnegative integers and positive integers respectively.

The **Virasoro algebra** Vir := Vir[$\mathbb{Z}$] (over $\mathbb{C}$) is the Lie algebra with the basis $\{C, d_i \mid i \in \mathbb{Z}\}$ and the Lie brackets defined by

$$[d_m, d_n] = (n - m)d_{m+n} + \delta_{m,-n}\tfrac{m^3-m}{12}C, \qquad \forall m, n \in \mathbb{Z},$$

$$[d_m, C] = 0, \qquad \forall m \in \mathbb{Z}.$$

The structure theory of Harish-Chandra modules over the Virasoro algebra is developed fairly well. For details, we refer the readers to [Ma], [MP], [MZ], the book [KR] and the references therein. In particular, the classification of irreducible Harish-Chandra modules was obtained in [Ma], while indecomposable modules was studied in [MP]. This classification was recently used to give the classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra [LZ2].

Patera and Zassenhaus [PZ] introduced the **generalized Virasoro algebra** Vir[G] for any additive subgroup G of $\mathbb{C}$ from the context of mathematics and physics. This Lie algebra can be obtained from Vir by replacing the index group $\mathbb{Z}$ with G (see Definition 2.1). This Lie algebra Vir[G] is called a **rank n Virasoro algebra** (or a **higher rank Virasoro algebra** if $n \geq 2$) if $G \simeq \mathbb{Z}^n$.

Representation theory of generalized Virasoro algebras have been extensively studied in recent years. Mazorchuk [M] proved that all irreducible Harish-Chandra modules over Vir[$\mathbb{Q}$] are intermediate series modules (where $\mathbb{Q}$ is the field of rational numbers). In [HWZ],

a criterion for the irreducibility of Verma modules over the generalized Virasoro algebra Vir$[G]$ was obtained. In [S1], [S2] and [SZ], the irreducible Harish-Chandra modules over the generalized Virasoro algebras were investigated. [BZ] constructed a new class of irreducible Harish-Chandra modules over some generalized Virasoro algebras. Recently, a complete classification of irreducible Harish-Chandra modules over higher rank Virasoro algebras was given by the last two authors of the present paper [LZ1].

In this paper, we give the classification of irreducible Harish-Chandra modules over any generalized Virasoro algebra.

The paper is organized as follows. In Section 2, for the reader's convenience, we collect some results and give some notations from [M], [MP], [SZ] and [LZ1] for later use. In Section 3, we widely use the results in [M], [LZ1] and [SZ] to complete the classification of irreducible weight modules with finite dimensional weight spaces over Vir$[G]$. The classification strongly depends on the index group G. If G does not have a direct summand $\mathbb{Z}$, then such irreducible modules over Vir$[G]$ are only modules of intermediate series whose weight spaces are all 1-dimensional. Otherwise, there is one more class of modules which are constructed by using intermediate series modules over a generalized Virasoro subalgebra Vir$[G_0]$ of Vir$[G]$ for a direct summand G_0 of G with corank 1 (Theorem 3.8). We break the proof into seven lemmas.

Throughout this paper, a subgroup always means an additive subgroup if not specified. For any $a \in \mathbb{C}$ and $S \subset \mathbb{C}$, we denote $a + S = \{a + x | x \in S\}$ and $aS = Sa = \{ax | x \in S\}$.

§2. Modules over generalized Virasoro algebras

First we give the definition of the generalized Virosoro algebras.

Definition 2.1. *Let G be a nonzero additive subgroup of $\mathbb{C}$. The* **generalized Virasoro algebra** Vir$[G]$ *(over $\mathbb{C}$) is the Lie algebra with the basis $\{C, d_x \mid x \in G\}$ and the Lie bracket defined by*

$$[d_x, d_y] = (y - x)d_{x+y} + \delta_{x,-y}\frac{x^3 - x}{12}C, \qquad \forall\ x, y \in G,$$

$$[C, d_x] = 0,\ \forall\ x \in G.$$

It is clear that Vir$[G] \cong$ Vir$[aG]$ for any $a \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Then for any $x \in G^* := G\backslash\{0\}$, Vir$[x\mathbb{Z}]$ is a Lie subalgebra of Vir$[G]$ isomorphic to Vir $=$ Vir$[\mathbb{Z}]$, the classical Virasoro algebra.

A Vir$[G]$-module V is called **trivial** if Vir$[G]V = 0$. For any Vir$[G]$-module V and $c, \lambda \in \mathbb{C}$, $V_{\lambda,c} := \{v \in V \mid d_0 v = \lambda v, Cv = cv\}$ is called the **weight space** of V corresponding to the weight (λ, c). When C acts as a scalar c on the whole module V, we shall simply write V_λ instead of $V_{\lambda,c}$. In the rest of this section, all modules considered are such modules.

A Vir$[G]$-module V is called a **weight module** if V is the sum of its weight spaces, and a weight module is called a **Harish-Chandra module** if all the weight spaces are finite dimensional. For a weight module V, we define supp$V := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$, which is generally called the **weight set** (or the **support**) of V. Given a weight module V and any subset $S \subset \mathbb{C}$, we denote $V_S = \bigoplus_{x \in S} V_x$, where $V_x = 0$ for $x \notin \mathrm{supp}V$.

Let V be a module and $W' \subset W$ are submodules of V. The module W/W' is called a **sub-quotient** of V. If $W' = 0$ we consider that $W = W/W'$.

Let V be a weight module over Vir$[G]$. V is said to be **uniformly bounded**, if there exists $N \in \mathbb{N}$ such that $\dim V_x < N$ for all $x \in \mathrm{supp}V$.

Fix a total order "$\succeq$" on G which is compatible with the addition, i.e., $x \succeq y$ implies $x + z \succeq y + z$ for any $x, y, z \in G$. Let

$$G^+ := \{x \in G \mid x \succ 0\}, \quad G^- := \{x \in G \mid x \prec 0\},$$

$$\mathrm{Vir}[G]^+ := \sum_{x \in G^+} \mathbb{C} d_x, \quad \mathrm{Vir}[G]^- := \sum_{x \in G^-} \mathbb{C} d_x.$$

Let V be a weight module over Vir$[G]$. A vector $v \in V_{\lambda,c}$, $\lambda \in \mathrm{supp}V$, $c \in \mathbb{C}$, is called a **highest weight (resp. lowest weight) vector** if $\mathrm{Vir}[G]^+ v = 0$ (resp. $\mathrm{Vir}[G]^- v = 0$). V is called a **highest weight (resp. lowest weight) module** with highest weight (resp. lowest weight) (λ, c) if there exists a nonzero highest (lowest, resp.) weight vector $v \in V_{\lambda,c}$ such that V is generated by v. For the natural total order on $\mathbb{Z}$, the irreducible highest weight Vir$[\mathbb{Z}]$-module with highest weight (λ, c) is generally denoted by $V(c, \lambda)$.

Now we give another class of weight modules over Vir$[G]$, i.e., the **modules of intermediate series** $V(\alpha, \beta, G)$. For any $\alpha, \beta \in \mathbb{C}$, the module $V(\alpha, \beta, G)$ has a basis $\{v_x \mid x \in G\}$ with actions of Vir$[G]$ given by:

$$Cv_y = 0, \quad d_x v_y = (\alpha + y + x\beta) v_{x+y}, \ \forall\, x, y \in G.$$

One knows from [SZ] that $V(\alpha, \beta, G)$ is reducible if and only if $\alpha \in G$ and $\beta \in \{0, 1\}$. By $V'(\alpha, \beta, G)$ we denote the unique nontrivial irreducible sub-quotient of $V(\alpha, \beta, G)$. Then

$\mathrm{supp}(V'(\alpha,\beta,G)) = \alpha + G$ or $\mathrm{supp}(V'(\alpha,\beta,G)) = G \setminus \{0\}$. We also refer $V'(\alpha,\beta,G)$ as **intermediate series modules**. The following result is due to Su and Zhao.

Theorem 2.2 ([SZ, Theorem 4.6]). *Let V be a nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *with all weight spaces* 1*-dimensional. Then $V \cong V'(\alpha,\beta,G)$ for some $\alpha,\beta \in \mathbb{C}$.*

This result for the classical Virasoro algebra is due to Kaplansky [K1]. The following classification of irreducible Harish-Chandra modules over the classical Virasoro algebra was obtained by Mathieu.

Theorem 2.3 ([Ma]). *Every irreducible Harish-Chandra module over* Vir *is either a highest weight module, a lowest weight module, or a module of intermediate series.*

We say that a $\mathrm{Vir}[\mathbb{Z}b]$-module W is **positively truncated** (**negatively truncated**) relative to b if for any $\lambda \in \mathrm{supp}V$, there exists some $x_0 \in \mathbb{Z}$ such that $\mathrm{supp}(V_{\lambda+\mathbb{Z}b}) \subset \{\lambda + xb | x \leq x_0\}$ (resp. $\mathrm{supp}(V_{\lambda+\mathbb{Z}b}) \subset \{\lambda + xb | x \geq x_0\}$). The following result from Martin and Piard will be useful to our later proofs.

Theorem 2.4 ([MP]). *Every Harish-Chandra* Vir*-module $V[\mathbb{Z}b]$, $b \in \mathbb{C}$ which has neither trivial submodules nor trivial quotient modules can be decomposed as a direct sum of three submodules $V = V^+ \oplus V^0 \oplus V^-$, where V^+ is positively truncated, V^- is negatively truncated and V^0 is uniformly bounded.*

Now we come back to the generalized Virasoro algebras. Marzorchuck proved

Theorem 2.5 ([M]). *Any nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[\mathbb{Q}]$ *is a module of intermediate series.*

In fact, he has proved the following theorem as he remarked at the end of his paper [M]:

Theorem 2.5′. *Let G be an infinitely generated additive subgroup of $a\mathbb{Q}$ for some $a \in \mathbb{C}$. Then any nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is a module of intermediate series.*

Now we assume that $G = G_0 \oplus \mathbb{Z}b \subset \mathbb{C}$ where $0 \neq b \in \mathbb{C}$ and G_0 is a nonzero subgroup of $\mathbb{C}$. (Note that some G does not possess this property, for example, $\mathbb{Q}$). Set

$$\mathrm{Vir}[G]_+ = \bigoplus_{x \in G_0, k \in \mathbb{Z}^+} \mathbb{C}d_{x+kb} \oplus \mathbb{C}C$$

and

$$\mathrm{Vir}[G]_{++} = \bigoplus_{x\in G_0, k\in \mathbb{N}} \mathbb{C} d_{x+kb}.$$

Given any $\alpha, \beta \in \mathbb{C}$, let $V'(\alpha, \beta, G_0)$ be the module of intermediate series over $\mathrm{Vir}[G_0]$. We extend the $\mathrm{Vir}[G_0]$-module structure on $V'(\alpha, \beta, G_0)$ to a $\mathrm{Vir}[G]_+$-module structure by defining $\mathrm{Vir}[G]_{++}V'(\alpha, \beta, G_0) = 0$. Then we obtain the induced $\mathrm{Vir}[G]$-module

$$M(b, G_0, V'(\alpha, \beta, G_0)) = U(\mathrm{Vir}[G]) \otimes_{U(\mathrm{Vir}[G]_+)} V'(\alpha, \beta, G_0),$$

where $U(\mathrm{Vir}[G])$ and $U(\mathrm{Vir}[G]_+)$ are the universal enveloping algebras of $\mathrm{Vir}[G]$ and $\mathrm{Vir}[G]_+$, respectively.

The $\mathrm{Vir}[G]$-module $M(b, G_0, V'(\alpha, \beta, G_0))$ has a unique maximal proper submodule J. Then we obtain the irreducible quotient module

$$V(\alpha, \beta, b, G_0) := M(b, G_0, V'(\alpha, \beta, G_0))/J.$$

It is clear that this module is uniquely determined by α, β, b and G_0 and that ([Lemma 3.8, LZ1])

$$\mathrm{supp}V(\alpha, \beta, b, G_0) = \ \alpha - \mathbb{Z}^+ b + G_0 \quad \text{or} \quad (-\mathbb{Z}^+ b + G_0) \setminus \{0\}.$$

It was proved that $V(\alpha, \beta, b, G_0)$ is a Harish-Chandra module over $\mathrm{Vir}[G]$:

Theorem 2.6 ([BZ, Theorem 3.1]). *The* $\mathrm{Vir}[G]$*-modules* $V(\alpha, \beta, b, G_0)$ *are Harish-Chandra modules. More precisely,* $\dim V_{-ib+\alpha+x} \le (2i+1)!!$ *for all* $i \in \mathbb{N}$, $x \in G_0$.

From this theorem, we easily deduce the following corollary which will be used frequently in our later proofs.

Corollary 2.7. *Let* $V = V(\alpha, \beta, b, G_0)$ *and* $i \in \mathbb{Z}^+$. *Then for any subgroup* G' *of* G, *the* $\mathrm{Vir}[G']$ *module* $V_{\alpha-ib+G'}$ *is uniformly bounded if and only if* $G' \subset G_0$.

The classification of irreducible Harish-Chandra modules over higher rank Virasoro algebra was obtained by the last two author of the present paper.

Theorem 2.8 ([LZ1]). *Let* G *be an additive subgroup of* $\mathbb{C}$ *such that* $G \cong \mathbb{Z}^n$ *for some* $n \in \mathbb{N}$ *with* $n > 1$. *Then any nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is either of intermediate series or isomorphic to some* $V(\alpha, \beta, G_0, b)$ *for some* $\alpha, \beta \in \mathbb{C}$, $b \in G$ *and a subgroup* G_0 *of* G *such that* $G = G_0 \oplus \mathbb{Z}b$.

§3. Classification of irreducible Harish-Chandra modules over generalized Virasoro algebras

In this section we give a classification of the irreducible Harish-Chandra modules over the generalized Virasoro algebras. Let us proceed with the convention that $U(G) = U(\mathrm{Vir}[G])$, the enveloping algebra, and that $U(G)_a = \{y \in U(\mathrm{Vir}[G]) \,|\, [d_0, y] = ay\}, a \in \mathbb{C}$, for any additive subgroup G of $\mathbb{C}$.

We recall the concept: the rank of a subgroup A of $\mathbb{C}$ from [K2]. The **rank** of A, denoted by $\mathrm{rank}(A)$, is the maximal number r with $g_1, \cdots, g_r \in A \setminus \{0\}$ such that $\mathbb{Z} g_1 + ... + \mathbb{Z} g_r$ is a direct sum. If such an r does not exist, we define $\mathrm{rank}(A) = \infty$.

From now on we fix a subgroup G of $\mathbb{C}$, and a nontrivial irreducible Harish-Chandra module V over $\mathrm{Vir}[G]$. Because of the classifications in Theorems 2.2 and 2.5′, we may also **assume that rank$G \geq 2$.**

Lemma 3.1. *For any finite subset I of supp$V$, there is a subgroup G_I of G such that*

(a) $G_I \cong \mathbb{Z}^k$ *for some* $k \in \mathbb{N}$,

(b) $U(G_I)V_\mu = U(G_I)V_{\mu'}$ *and* $\mu - \mu' \in G_I$ *for any* $\mu, \mu' \in I$,

(c) V_μ *is an irreducible* $U(G_I)_0$ *module for any* $\mu \in I$.

Proof. Assume that $I = \{\mu_i | i = 1, 2, ..., s\}$. Since V is an irreducible $U(G)$-module and V_{μ_i} are finite dimensional, then for any $\mu_i, \mu_j \in I$, there are elements $y_{i,j}^{(1)}, ..., y_{i,j}^{(d_{ij})} \in U(G)_{\mu_i - \mu_j}$ such that $V_{\mu_i} = \sum_{t=1}^{d_{ij}} y_{i,j}^{(t)} V_{\mu_j}$, where $d_{ij} \in \mathbb{N}$.

For any $\mu_i \in \mathrm{supp}V$, since V is irreducible we see that V_{μ_i} is an irreducible $U(G)_0$-module. Let $\phi_{\mu_i} : U(G)_0 \longrightarrow gl(V_{\mu_i})$ be the representation of $U(G)_0$ in V_{μ_i}, where $gl(V_{\mu_i})$ is the general linear Lie algebra associated with the vector space V_{μ_i}. Since V_{μ_i} and hence $gl(V_{\mu_i})$ is finite dimensional, then $\phi_{\mu_i}(U(G)_0)$ is finite dimensional. Thus there exist $y_{i1}, ..., y_{im_i} \in U(G)_0$ such that $\mathrm{span}_{\mathbb{C}} \{\phi_{\mu_i}(y_{i1}), ..., \phi_{\mu_i}(y_{im_i})\} = \phi_{\mu_i}(U(G)_0)$.

For the finitely many elements: $y_{i1}, ..., y_{im_i}$, $y_{i,j}^{(1)}, ..., y_{i,j}^{(d_{ij})}$; $1 \leq i, j \leq s$, thanks to the PBW Theorem there are finitely many elements $g_1, ..., g_n \in G$ such that $y_{i,j}, y_{i,j}^{(k)} \in U(G_I)$, where G_I is the subgroup of G generated by $g_1, ..., g_n$.

Since G_I is a finitely generated torsion free abelian group, it is clear that $G_I \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$, (a) follows. By the construction of G_I, we know that $V_{\mu_i} \subset U(G_I)V_{\mu_j}$. Hence

$U(G_I)V_{\mu_i} = U(G_I)V_{\mu_j}$ for any $\mu_i, \mu_j \in I$. It is also clear that $\mu - \mu' \in G_I$ for any $\mu, \mu' \in I$. Thus (b) follows.

We now prove that V_{μ_i} is an irreducible $U(G_I)_0$-module for any $\mu_i \in I$. Suppose that N is a nontrivial $U(G_I)_0$-submodule of V_{μ_i}. For any element $y \in U(G)_0$, there are some $a_j \in \mathbb{C}$ such that $\phi_{\mu_i}(y) = \sum_{j=1}^{m_i} a_j \phi_{\mu_i}(y_{ij})$. Then $yN = \sum_{j=1}^{m_i} a_j \phi_{\mu_i}(y_{ij})N = (\sum_{j=1}^{m_i} a_j y_{ij})N \subset N$. That is, N is a nonzero $U(G)_0$-submodule of V_{μ_i}, forcing $N = V_{\mu_i}$. Hence, V_{μ_i} is an irreducible $U_0(G_I)$-module for any $\mu_i \in I$, and (c) follows. ∎

Lemma 3.2. *Let I and G_I be the same as in Lemma 3.1 and let G' be a subgroup of G that contains G_I. Then for any $\lambda \in I$, $V_{\lambda+G'}$ has a unique irreducible* Vir$[G']$*-sub-quotient V' with* $\dim V'_\mu = \dim V_\mu$, $\forall \mu \in I$.

Proof. Since $G_I \subset G'$, from (b) and (c) of Lemma 3.1, we know that,

(b′) $U(G')V_\mu = U(G')V_{\mu'}$ for any $\mu, \mu' \in I$, which we denote by W.

(c′) V_μ is an irreducible $U_0(G')$ module for any $\mu \in I$,

Clearly, $W \subset V_{\lambda+G'}$, $\forall \lambda \in I$.

Suppose that W' is a proper Vir$[G']$-submodule of W. For any $\mu \in I$, it is clear that W'_μ is a proper $U(G')_0$ submodule of W_μ, forcing $W'_\mu = 0$, that is, W' trivially intersects W_μ for any $\mu \in I$. Now let $V' = U(G')X/Y$, where $X = \oplus_{\mu \in I} V_\mu$ and Y is the sum of all $U(G')$-submodules Y' of $U(G')X$ such that $Y'_\mu = 0$ for all $\mu \in I$. Then V' is as desired. ∎

Theorem 3.3. (a) *If V is uniformly bounded, then V is of intermediate series.*

(b) *A Harish-Chandra module W over* Vir$[G]$ *with supp$(W) \subset \lambda + G$ for some $\lambda \in \mathbb{C}$ is uniformly bounded if and only if* $\dim W_\lambda = \dim W_\mu$, *for all $\lambda, \mu \in$ supp$V \setminus \{0\}$.*

Remark. This theorem holds also for any rank 1 group G.

Proof. (b) follows directly from (a). To prove (a), it suffices to show that $\dim V_\lambda = 1$, for all $\lambda \in \mathrm{supp}V$, by Theorem 2.2.

Now suppose that $\dim V_\lambda \geq 2$ for some $\lambda \in \mathrm{supp}V$. Using Lemma 3.1 for $I = \{\lambda\}$, we have the subgroup G_I of G described there. Then V_λ is an irreducible $U(G_I)_0$-module. Consider the uniformly bounded $Vir[G_I]$-module $V_{\lambda+G_I}$, then it has an irreducible uniformly bounded sub-quotient V' with $\dim V'_\lambda = \dim V_\lambda \geq 2$ since V_λ is an irreducible $U(G_I)_0$-module. by Theorem 2.8, $\dim V'_\lambda$ should be not larger than 1, a contradiction. Thus we have that

$dimV_\lambda = 1,\ \forall \lambda \in \mathrm{supp}V$. The proof is completed. ∎

Lemma 3.4. *For any $\lambda \in \mathrm{supp}V$ and $g \in G \setminus \{0\}$, if the $Vir[\mathbb{Z}g]$-module $V_{\lambda+\mathbb{Z}g}$ has a nontrivial uniformly bounded* $\mathrm{Vir}[\mathbb{Z}g]$*-sub-quotient, then $V_{\lambda+\mathbb{Z}g}$ itself is uniformly bounded.*

Proof. To the contrary, suppose that $V_{\lambda+\mathbb{Z}g}$ is not uniformly bounded. Then there exist some $\mu_1, \mu_2 \in \mathrm{supp}(V_{\lambda+\mathbb{Z}g}) \setminus \{0\}$ with $\dim V_{\mu_1} \neq \dim V_{\mu_2}$.

Applying Lemma 3.1 to $I = \{\mu_1, \mu_2\}$, we have the subgroup G_I of G described there, and furthermore we may assume that $\mathrm{rank}G_I > 1$. Let $G' = G_I + \mathbb{Z}g$. Then by Lemma 3.2, the $\mathrm{Vir}[G']$-module $V_{\lambda+G'}$ has a unique irreducible sub-quotient V' with $\dim V'_{\mu_i} = \dim V_{\mu_i}, i = 1, 2$.

Since $\dim V'_{\mu_1} \neq \dim V'_{\mu_2}$, Theorem 3.3 ensures that V' is not uniformly bounded. By Theorem 2.8, V' must be of the form $V(\alpha, \beta, G'_0, b)$, for some $\alpha, \beta \in \mathbb{C}$, $b \in G'$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b$. Since $\dim V'_{\mu_1} \neq \dim V'_{\mu_2}$ and $\mu_1 \neq 0 \neq \mu_2$, $V'_{\lambda+\mathbb{Z}g}$ is not uniformly bounded by Theorem 3.3. Then by Corollary 2.7, we know that $g \notin G'_0$. Thus the $\mathrm{Vir}[\mathbb{Z}g]$-module $V'_{\lambda+\mathbb{Z}g}$ is positively truncated relative to g.

From the definition of V' we know that $V'_{\lambda+\mathbb{Z}g}$ is a $\mathrm{Vir}[\mathbb{Z}g]$-sub-quotient of $V_{\lambda+\mathbb{Z}g}$, say, $V'_{\lambda+\mathbb{Z}g} = W/W'$ where $W' \subset W$ are $\mathrm{Vir}[\mathbb{Z}g]$-submodules of $V_{\lambda+\mathbb{Z}g}$. Since $\dim V_{\mu_1} = \dim V'_{\mu_1} = \dim W_{\mu_1}$, then $W'_{\mu_1} = 0$ and $(V_{\lambda+\mathbb{Z}g}/W)_{\mu_1} = 0$. Note that $\mu_1 \neq 0$. Therefore, $V_{\lambda+\mathbb{Z}g}/W$, W/W', W' all do not have uniformly bounded $\mathrm{Vir}[\mathbb{Z}g]$-sub-quotient. Thus $V_{\lambda+\mathbb{Z}g}$ does not have any nontrivial uniformly bounded $\mathrm{Vir}[\mathbb{Z}g]$-sub-quotient, a contradiction. ∎

Lemma 3.5. *Assume that $\lambda \in \mathrm{supp}V \setminus \{0\}$ and that G_1, G_2 are any subgroups of G. If both $V_{\lambda+G_1}$ and $V_{\lambda+G_2}$ are uniformly bounded (as* $\mathrm{Vir}[G_1]$*-module and* $\mathrm{Vir}[G_2]$*-modules, resp.), then $V_{\lambda+G_1+G_2}$ is a uniformly bounded* $\mathrm{Vir}[G_1+G_2]$*-module.*

Proof. Thanks to Theorem 3.3, we may assume that $\dim V_\mu = m$ for nonzero $\mu \in (\lambda + G_1) \cup (\lambda + G_2)$.

To the contrary, we suppose that there are some $g_i \in G_i, i = 1, 2$ such that $\dim V_{\lambda+g_1+g_2} \neq m$ with $(\lambda+g_1)(\lambda+g_2)(\lambda+g_1+g_2) \neq 0$. Applying Lemma 3.1 to $I = \{\lambda+g_1, \lambda+g_2, \lambda+g_1+g_2\}$, we have the subgroup G_I as described in Lemma 3.1, and furthermore we may assume that $\mathrm{rank}G_I > 1$. Note that $g_1, g_2 \in G_I$. Lemma 3.2 ensures that the $\mathrm{Vir}[G_I]$-module $V_{\lambda+G_I}$ has a unique irreducible sub-quotient V' such that $\dim V'_\mu = \dim V_\mu$ for all $\mu \in I$. In particular, $\dim V'_{\lambda+g_1} \neq \dim V_{\lambda+g_1+g_2}$. Noting that $(\lambda + g_1)(\lambda + g_1 + g_2) \neq 0$, by Theorem

3.3 we know that V' is not uniformly bounded. By Theorem 2.8, $V' \cong V(\alpha, \beta, G_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G_I$ and a subgroup G_0 of G_I with $G_I = G_0 \oplus \mathbb{Z}b$. The fact that $V'_{\lambda+\mathbb{Z}g_1}$ and $V'_{\lambda+\mathbb{Z}g_2}$ are both uniformly bounded implies $g_1, g_2 \in G_0$. Thus $g_1, g_2, g_1+g_2 \in G_0$. By Corollary 2.7 , we deduce that $V'_{\lambda+\mathbb{Z}g_1+\mathbb{Z}g_2}$ is uniformly bounded. Then by Theorem 3.3, we see that $\dim V_{\lambda+g_1+g_2} = \dim V_{\lambda+g_1} = m$, a contradiction. Hence, $\dim V_{\lambda+x} = m$ for any $x \in G_1 + G_2$ with $\lambda + x \neq 0$. Therefore $V_{G_1+G_2}$ is uniformly bounded. ∎

Lemma 3.6. *For any $\mu \in \mathrm{supp}V \setminus \{0\}$, there exists a unique maximal subgroup G_μ of G such that $V_{\mu+G_\mu}$ is a uniformly bounded $\mathrm{Vir}[G_\mu]$-module. Furthermore,*

(a) *$G_{\mu_1} = G_{\mu_2}$ for any $\mu_1, \mu_2 \in \mathrm{supp}V \setminus \{0\}$, which we denote by $G^{(0)}$,*

(b) *either $G^{(0)} = G$ or $G \cong G^{(0)} \oplus \mathbb{Z}b$ for some $b \in G$.*

Proof. The existence and uniqueness of G_μ for $\mu \in \mathrm{supp}V \setminus \{0\}$ follows easily from Lemma 3.5.

(a) Fix $\mu_1 \neq \mu_2 \in \mathrm{supp}V \setminus \{0\}$ and $g_1 \in G_{\mu_1} \setminus \{0\}$. Let $I = \{\mu_1, \mu_2\}$ and take G_I the same subgroup described as in lemma 3.1, and furthermore we may assume that $\mathrm{rank}G_I > 1$. Set $G' = G_I + \mathbb{Z}(\mu_1 - \mu_2) + \mathbb{Z}g_1$. By Lemma 3.2, the $\mathrm{Vir}[G']$-module $V_{\mu_2+G'}$ has a unique irreducible sub-quotient V' with $\dim V'_{\mu_i} = \dim V_{\mu_i}, i = 1, 2$. Clearly, V' is nontrivial.

If V' is uniformly bounded, then $V'_{\mu_2+\mathbb{Z}g_1}$ is a uniformly bounded $\mathrm{Vir}[\mathbb{Z}g_1]$-sub-quotient of $V_{\mu_2+\mathbb{Z}g_1}$.

If V' is not uniformly bounded, then as a $\mathrm{Vir}[G']$-module, $V' \cong V(\alpha, \beta, G'_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G'$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b$. Note that $\mu_1 + \mathbb{Z}g_1 \subset \mu_2 + G'$. Since $V'_{\mu_1+\mathbb{Z}g_1}$ is uniformly bounded, then $g_1 \in G'_0$, and hence $V'_{\mu_2+\mathbb{Z}g_1}$ is also uniformly bounded by Corollary 2.7.

Thus in both cases, $V'_{\mu_2+\mathbb{Z}g_1}$ is a uniformly bounded $\mathrm{Vir}[\mathbb{Z}g_1]$-sub-quotient of $V_{\mu_2+\mathbb{Z}g_1}$. By Lemma 3.4, $V_{\mu_2+\mathbb{Z}g_1}$ is a uniformly bounded $\mathrm{Vir}[\mathbb{Z}g_1]$-module, forcing $g_1 \in G_{\mu_2}$ by Lemma 3.5.

So $G_{\mu_1} \subset G_{\mu_2}$. Symmetrically, we also have $G_{\mu_2} \subset G_{\mu_1}$. Thus $G_{\mu_1} = G_{\mu_2}, \forall \mu_1, \mu_2 \in \mathrm{supp}V \setminus \{0\}$.

(b) Suppose $G^{(0)} \neq G$. We shall prove that $G \cong G^{(0)} \oplus \mathbb{Z}b$ for some $b \in G$ in three steps.

Step 1: *$G/G^{(0)}$ is torsion-free.*

Otherwise we may choose some $g \in G \setminus G^{(0)}$ and $k_0 \in \mathbb{N}$ such that $k_0 g \in G^{(0)}$. Take any $\mu \in \mathrm{supp}V \setminus \{0\}$. Since $\mathrm{rank}G > 1$ we have a subgroup A of G such that $A \cong \mathbb{Z}^2$. By considering the nontrivial Vir$[A]$-module $V_{\mu+A}$ and using Theorem 2.8, we know that there exists $\lambda \in \mathrm{supp}(V) \setminus \mathbb{Z}g$.

Then $V_{\lambda+\mathbb{Z}g}$ is not uniformly bounded while $V_{\lambda+\mathbb{Z}k_0g}$ is uniformly bounded, by the definition of $G^{(0)}$. Since $V_{\lambda+\mathbb{Z}g}$ is not uniformly bounded, it has a non-trivial highest or lowest weight irreducible subquotiens, say non-trivial highest weight Vir$[\mathbb{Z}g]$-sub-quotient W. Without loss of generality, we may assume that W has the highest weight $\lambda \neq 0$. Then $W_{\lambda+\mathbb{Z}k_0g}$ has a nontrivial highest weigh sub-quotient. Thus $W_{\lambda+\mathbb{Z}k_0g}$ is not uniformly bounded, contradicting the fact that $V_{\lambda+\mathbb{Z}k_0g}$ is uniformly bounded. Thus $G/G^{(0)}$ is torsion-free. Step 1 follows.

Step 2: $\mathrm{rank}(G/G^{(0)}) = 1$**.**

Otherwise we assume that $g_1, g_2 \in G \backslash G^{(0)}$ are independent modulo $G^{(0)}$, i.e., the subgroup $G' = \langle g_1, g_2 \rangle$ is isomorphic to $\mathbb{Z}^2$. Then $G' \cap G^{(0)} = 0$. Take any $\lambda \in \mathrm{supp}V \setminus \{0\}$ and consider the Vir$[G']$-module $V_{\lambda+G'}$, which has an irreducible sub-quotient V' with $V'_\lambda \neq 0$.

If V' is uniformly bounded, then $V'_{\lambda+\mathbb{Z}g_1}$ is a nontrivial uniformly bounded Vir$[\mathbb{Z}g_1]$-sub-quotient of $V_{\lambda+\mathbb{Z}g_1}$. Thus by Lemma 3.4, $V_{\lambda+\mathbb{Z}g_1}$ is a uniformly bounded Vir$[\mathbb{Z}g_1]$ module, forcing $g_1 \in G^{(0)}$ by Lemma 3.5, a contradiction.

If V' is not uniformly bounded, then $V' \cong V(\alpha, \beta, G'_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G'$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b$. Then $V'_{\lambda+\mathbb{Z}g}$ is uniformly bounded for any $g \in G'_0 \setminus \{0\}$, and hence $V_{\lambda+\mathbb{Z}g}$ is also uniformly bounded. Thus $g \in G^{(0)}$, contradicting the fact that $G' \cap G^{(0)} = \{0\}$. Step 2 follows.

Thus $G \subset G^{(0)} + \mathbb{Q}g$ for any $g \in G \setminus G^{(0)}$.

Fix $g \in G \setminus G^{(0)}$, we know that $G \subset G^{(0)} + \mathbb{Q}g$. Since $G/G^{(0)}$ is torsion-free, $G^{(0)} \cap \mathbb{Q}g = \{0\}$. Then $G = G^{(0)} \oplus G_1$, where $G_1 = G \cap \mathbb{Q}g$.

Step 3: $G_1 \cong \mathbb{Z}$**.**

Otherwise, G_1 would be an infinitely generated abelian group of rank 1. Then by Theorem 2.5$'$, the Vir$[G_1]$ module $V_{\lambda+G_1}$ is uniformly bounded for any $\lambda \in \mathrm{supp}V$, forcing $G_1 \subset G^{(0)}$, contradiction again. Thus we must have $G_1 \cong \mathbb{Z}$. This complete the proof. ∎

Theorem 3.7. *Suppose that V is an irreducible Harish-Chandra module over the gen-*

eralized Virasoro algebra $\mathrm{Vir}[G]$ *that is not uniformly bounded. Then* V *is isomorphic to* $V(\alpha, \beta, G^{(0)}, b)$ *for some* $\alpha, \beta \in \mathbb{C}$, $b \in G$ *and a subgroup* $G^{(0)}$ *of* G *with* $G = G^{(0)} \oplus \mathbb{Z}b$.

Proof. Note that we have assumed $\mathrm{rank} G \geq 2$. Since V is not uniformly bounded, we use Lemma 3.6 to have $G^{(0)}$ and b as described there. Then C trivially acts on V.

Take any $\mu \in \mathrm{supp} V \setminus \{0\}$. Since $\mathrm{rank} G > 1$ we have a subgroup A of G such that $A \cong \mathbb{Z}^2$. By considering the nontrivial $\mathrm{Vir}[A]$-module $V_{\mu+A}$ and using Theorem 2.8, we know that there exists $\lambda \in \mathrm{supp}(V) \setminus \mathbb{Z}g$. The $\mathrm{Vir}[\mathbb{Z}b]$-module $W = V_{\lambda+\mathbb{Z}b}$ cannot have any nontrivial uniformly bounded sub-quotient, for otherwise we would have $b \in G^{(0)}$ by Lemma 3.5, contradicting the definition of $G^{(0)}$. Note that $0 \notin \mathrm{supp} W$. Then by Theorem 2.4, $W = W^+ \oplus W^-$, where W^+ is such that $\mathrm{supp} W^+ \subset \{\lambda + kb | k \leq t_0\}$ for some $t_0 \in \mathbb{Z}$ and W^- is such that $\mathrm{supp} W^- \subset \{\lambda + kb | k \geq s_0\}$ for some $s_0 \in \mathbb{Z}$.

Since $0 \notin \lambda + \mathbb{Z}b$, it is clear that the highest weight $\lambda + k_1 b$ of any irreducible highest weight $\mathrm{Vir}[\mathbb{Z}b]$-sub-quotient of W must satisfy $k_1 \leq t_0$ and that the lowest weight $\lambda + k_2 b$ of any irreducible lowest weight $\mathrm{Vir}[\mathbb{Z}b]$ sub-quotient of W must satisfy $k_2 \geq s_0$.

We want to show that one of W^+ and W^- is zero. Otherwise we may choose $t > t_0$ and $s < s_0$ such that both $\lambda + tb$ and $\lambda + sb$ lie in $\mathrm{supp} W$. Let $I = \{\lambda + tb, \lambda + sb\}$, and take G_I as in Lemma 3.1, and furthermore we may assume that $\mathrm{rank} G_I > 1$. Set $G' = G_I + \mathbb{Z}b$. The $\mathrm{Vir}[G']$-module $V_{\lambda+G'}$ has a unique irreducible sub-quotient V' with $\dim V'_\mu = \dim W_\mu = \dim V_\mu$ for all $\mu \in I$.

Clearly, V' is not a uniformly bounded $\mathrm{Vir}[G']$-module. Now by Theorem 2.8, we have that $V' \cong V(\alpha, \beta, G'_0, \epsilon b)$ for some $\alpha, \beta \in \mathbb{C}$, $\epsilon \in \{1, -1\}$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b$. Thus, $V'_{\lambda+\mathbb{Z}b}$ is either a positively truncated or a negatively truncated $\mathrm{Vir}[\mathbb{Z}b]$-module, and in both cases $\dim V'_{\lambda+tb}$ and $\dim V'_{\lambda+sb}$ are both nonzero. This implies that either $W = V_{\lambda+\mathbb{Z}b}$ has a highest weight $\mathrm{Vir}[\mathbb{Z}b]$ sub-quotient with highest weight $\lambda + kb$ with $k \geq t > t_0$ or has a lowest weight $\mathrm{Vir}[\mathbb{Z}b]$ sub-quotient with lowest weight $\lambda + kb$ with $k \leq s < s_0$, contradiction.

Hence, $W = W^+$ or $W = W^-$. Without loss of generality, we assume that $W = W^+$, that is, $\dim V_{\lambda+kb} = 0, \forall k > t_0$. But $V_{\lambda+kb+G^{(0)}}$ is a uniformly bounded $\mathrm{Vir}[G^{(0)}]$-module for any $k \in \mathbb{Z}$, then we must have that $\dim V_{\lambda+kb+g} = \dim V_{\lambda+kb} = 0, \forall k > t_0, g \in G^{(0)}$, provided that $\lambda + kb + g \neq 0$.

Let t_1 be the largest integer such that $\dim V_{\lambda+t_1 b} \neq 0$.

If $0 \in \lambda + t_1 b + G^{(0)}$, then $\lambda + t_1 b + G^{(0)} = G^{(0)}$. Then $\mathrm{supp}(V) \subset -\mathbb{Z}^+ b + G^{(0)}$. Any irreducible $\mathrm{Vir}[G^{(0)}]$-submodule W of $V_{G^{(0)}}$ generates V as $\mathrm{Vir}[G]$-module. Using PBW theorem we know that $W = V_{G^{(0)}}$. Since $V_{G^{(0)}} \supset V_{\lambda + t_1 b} \neq 0$, $V_{G^{(0)}}$ is a uniformly bounded nontrivial irreducible $\mathrm{Vir}[G^{(0)}]$-module. So $V_{G^{(0)}} \cong V'(\alpha, \beta, G^{(0)})$ for some $\alpha, \beta \in \mathbb{C}$. Consequently $V \cong V(\alpha, \beta, G^{(0)}, b)$.

If $0 \notin \lambda + t_1 b + G^{(0)}$, and $0 \notin \lambda + (t_1 + 1)b + G^{(0)}$ or $0 \in \lambda + (t_1 + 1)b + G^{(0)}$ but $V_0 = 0$, then $\mathrm{Vir}_{b+G^{(0)}} V_{G^{(0)}} = 0$ where $\mathrm{Vir}_{b+G^{(0)}} = \sum_{x \in b+G^{(0)}} \mathbb{C} d_x$. So $\mathrm{Vir}_{\mathbb{N} b+G^{(0)}} V_{G^{(0)}} = 0$. Thus $\mathrm{supp}(V) \subset -\mathbb{Z}^+ b + G^{(0)}$. Similar discussions yield to the same conclusion that $V \cong V(\alpha, \beta, G^{(0)}, b)$.

If $0 \in \lambda + (t_1 + 1)b + G^{(0)}$, and $V_0 \neq 0$, take nonzero $v \in V_0$. Then $\mathrm{Vir}_{b+G^{(0)}} v = 0$ and $\mathrm{Vir}_{G^{(0)}} v = 0$. Thus $\mathrm{Vir}_{\mathbb{Z}^+ b+G^{(0)}} v = 0$. Since V is not trivial, using PBW theorem and $C = 0$ we deduce that $V_{-\mathbb{Z} b+G^{(0)}}$ is a proper submodule, contradiction. So this case does not occur.

This proves the theorem. ∎

Combining Theorems 2.2, 2.5′, 3.3, and 3.7 we now have proved the following classification theorem.

Theorem 3.8. *Suppose that G is an arbitrary additive subgroup of $\mathbb{C}$.*

(a) *If* $\mathrm{rank} G = 1$ *and* $G \not\cong \mathbb{Z}$, *then any nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is a module of the intermediate series.*

(b) *If* $G \cong \mathbb{Z}$, *then any nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is a module of the intermediate series, a highest weigh module or a lowest weight module.*

(c) *If* $\mathrm{rank} G > 1$, *then any nontrivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is either a module of the intermediate series or isomorphic to* $V(\alpha, \beta, G^{(0)}, b)$ *for some* $\alpha, \beta \in \mathbb{C}$, $b \in G$ *and a subgroup* $G^{(0)}$ *of* G *with* $G = G^{(0)} \oplus \mathbb{Z} b$.

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